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Universal periods of hyperelliptic curves and their applications

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Abstract

We construct universal power series for differential 1-forms and period integrals of Schottky–Mumford uniformized hyperelliptic curves over local fields. Using these universal 1-forms and periods, we characterize Siegel modular forms vanishing on the hyperelliptic Jacobian locus, and construct universal and p-adic solutions of the Korteweg–de Vries hierarchy.

MSC: 14H25; 14H42; 14H45; 35Q53

0. Introduction

In this paper, we would like to note that observations in [8,9] are applicable to the hyperelliptic case, using a result of Gerritzen–van der Put [6] on the Schottky–Mumford uniformization of hyperelliptic curves. More precisely, we will construct *universal power series* for differential 1-forms and period integrals of certain hyperelliptic curves over (archimedean and nonarchimedean) local fields, and will give their applications as follows:

1. To characterize Siegel modular forms (over fields of characteristic $\neq 2$) vanishing on the hyperelliptic Jacobian locus in terms of certain relations between their Fourier coefficients.

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2. To construct a universal solution (deforming the soliton solution) of the Korteweg–de Vries (KdV) hierarchy, and p-adic solutions of KdV as specializations of this universal solution.

As for application 1, we note that there were results of Mumford [19] and Poor [21] on the hyperelliptic Schottky problem, however their approach, which characterizes periods of hyperelliptic curves in terms of the vanishing of certain theta constants, is different from ours. The solutions of KdV given in application 2 are constructed as universal and p-adic versions of the Riemann theta function solutions given by Novikov [20] and McKean–van Moerbeke [15].

Schottky uniformization theory with describing 1-forms and periods for algebraic curves over $\mathbf{C}$ was established by Schottky [22] (cf. [7,13]). The nonarchimedean version was constructed by Mumford [18] and Manin–Drinfeld [14], and further, Gerritzen–van der Put [6] gave the uniformization of hyperelliptic Mumford curves by certain Schottky groups called "Whittaker groups". In Section 1 using these results, we give a uniformization for hyperelliptic curves over local fields close to a degenerate curve $Y^2 = X\prod_{k=1}^{g}(X-\alpha_k^2)^2$. This uniformization, which is obtained from Whittaker groups with generators

$$\begin{pmatrix} \alpha_k & -\alpha_k \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & \beta_k^2 \end{pmatrix}\begin{pmatrix} \alpha_k & -\alpha_k \\ 1 & 1 \end{pmatrix}^{-1} \text{ modulo center,}$$

is useful in deforming the soliton solution because it is known to be expressed by the theta function of the above degenerate curve (cf. [19, Chapter IIIb, Section 5]). We note that this uniformization was used by Belokolos and others in [1, 5.8], for constructing the Riemann theta function solutions of KdV concretely. Our universal 1-forms and periods obtained in Section 2 are power series with polynomial coefficients over $\mathbf{Z}[1/2]$ which become, by specializing variables, the 1-forms and periods of hyperelliptic curves uniformized in this way (universal periods of hyperelliptic curves having reduction of another type were studied by Teitelbaum [23] in the genus 2 case). Therefore, as is described in Sections 3 and 4, one can obtain the hyperelliptic version (applications 1 and 2 above) of the results in [8,9] on the Schottky problem and constructing solutions of the KP hierarchy, respectively.

Lastly, we would like to mention Schottky uniformization theory on analytic curves of infinite genus over local fields (cf. [10]). This, combining the results in this paper, would yield a theory on hyperelliptic curves of infinite genus. It would be interesting to compare this approach with the well-known work of McKean and Trubowitz on "Hill's surfaces" (cf. [16,17]).

1. Uniformization of hyperelliptic curves

In this section, we recall Schottky uniformization theory on algebraic curves over local fields (cf. [22,18]), and construct a family of Schottky uniformized hyperelliptic curves using a result in [6]. Let K be $\mathbf{C}$ or a nonarchimedean complete valuation field with multiplicative valuation $|\ |$. Let $PGL_2(K)$ act on $\mathbf{P}^1(K)$ by the Möbius transformation:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}(z) = \frac{az+b}{cz+d}.$$

A subgroup Γ of $PGL_2(K)$ is a *Schottky group* of rank g over K if there exist (free) generators $\gamma_1, \ldots, \gamma_g$ of Γ and $2g$ open domains bounded by Jordan curves if $K = \mathbf{C}$ (resp. $2g$ open disks if K is a nonarchimedean valuation field) $D_{\pm 1}, \ldots, D_{\pm g} \subset \mathbf{P}^1(K)$ such that

$$\overline{D_i} \cap \overline{D_j} = \emptyset \ \ (i \neq j), \quad \gamma_k(\mathbf{P}^1(K) - D_{-k}) = \overline{D_k} \ \ (k = 1, \ldots, g),$$

where $\overline{D_i}$ denotes the closure of D_i. Put

$$F_\Gamma = \mathbf{P}^1(K) - \bigcup_{k=1}^{g} (D_k \cup \overline{D_{-k}}), \quad H_\Gamma = \bigcup_{\gamma \in \Gamma} \gamma(F_\Gamma).$$

Then it is easy to see that Γ acts freely and discontinuously on H_Γ, and $\mathbf{P}^1(K) - H_\Gamma$ becomes the limit set of Γ. Let C_Γ denote the quotient K-analytic space H_Γ/Γ which is obtained from $\mathbf{P}^1(K) - \bigcup_{k=1}^{g} D_{\pm k}$ identifying the boundaries ∂D_k and ∂D_{-k} via γ_k $(k = 1, \ldots, g)$. Then C_Γ is called *Schottky uniformized* by Γ. When $K = \mathbf{C}, C_\Gamma$ is a compact Riemann surface of genus g which becomes a (proper and smooth) algebraic curve over $\mathbf{C}$. Then for each $i = 1, \ldots, g$, let a_i be the closed path ∂D_i counterclockwise oriented, and let b_i be an oriented path in F_Γ from a point x_i of ∂D_{-i} to $\gamma_i(x_i)$ such that $b_i \cap b_j = \emptyset$ $(i \neq j)$. One can see that $\{a_i, b_i\}_{1 \leq i \leq g}$ becomes a canonical basis of $H_1(C_\Gamma, \mathbf{Z})$, so that

$$(a_i, b_j) = \delta_{ij}, \quad (a_i, a_j) = (b_i, b_j) = 0 \ \ (i, j \in \{1, \ldots, g\}).$$

When K is a nonarchimedean valuation field, it is shown in [18] (cf. [6, Chapter III]) that C_Γ can be algebraizable as a (proper and smooth) algebraic curve of genus g over K which we call a *Mumford curve.* Let

$$[a, b; c, d] = \frac{(a-c)(b-d)}{(a-d)(b-c)}$$

denote the cross ratio of four points a, b, c and d.

Theorem 1. (a) *Let $K = \mathbf{C}$, and take $\alpha_k, \beta_k \in K^\times$ $(k = 1, \ldots, g)$ such that $\alpha_i \neq \pm\alpha_j$ $(i \neq j)$ and that*

$$\left|\frac{\beta_k}{\alpha_k}\right|, \quad \left|\frac{\beta_k}{\alpha_i \pm \alpha_j}\right| \quad (i, j \neq k)$$

are sufficiently small. Then the subgroup $\Gamma \subset PGL_2(K)$ generated by $\gamma_1, \ldots, \gamma_g$:

$$\gamma_k = \begin{pmatrix} \alpha_k & -\alpha_k \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \beta_k^2 \end{pmatrix} \begin{pmatrix} \alpha_k & -\alpha_k \\ 1 & 1 \end{pmatrix}^{-1} \bmod (K^\times),$$

becomes a Schottky group of rank g, and C_Γ is a hyperelliptic curve of genus g over $K = \mathbf{C}$.

(b) *Let K be a nonarchimedean complete valuation field of characteristic $\neq 2$, and take $\alpha_k, \beta_k \in K^\times$ $(k = 1,\dots,g)$ such that $\alpha_i \neq \pm\alpha_j$ $(i \neq j)$ and that*

$$|\beta_k|^2 < \min\{|[\alpha_k, -\alpha_k; \pm\alpha_i, \pm\alpha_j]|;\ i,j \neq k\} \quad (k = 1,\dots,g).$$

Then the $\gamma_k \in PGL_2(K)$ $(k = 1,\dots,g)$ defined as above generate a Schottky group Γ of rank g, and C_Γ is a hyperelliptic curve of genus g over K.

(c) *In cases* (a) *and* (b), *the affine equation of C_Γ is given by*

$$Y^2 = X\prod_{k=1}^{g}(X - \theta(\lambda_k))(X - \theta(\mu_k)),$$

where

$$\theta(z) = z^2 \prod_{\gamma\in\Gamma-\{1\}} \left(\frac{z-\gamma(0)}{z-\gamma(\infty)}\right)^2$$

and

$$\lambda_k = \alpha_k\frac{1-\beta_k}{1+\beta_k}, \quad \mu_k = \alpha_k\frac{1+\beta_k}{1-\beta_k}.$$

Further, under $\beta_1,\dots,\beta_g \to 0$, C_Γ tends to the degenerate curve obtained from $\mathbf{P}^1_K$ by identifying α_k and $-\alpha_k$ $(k = 1,\dots,g)$ in pairs, of which affine equation is given by

$$Y^2 = X\prod_{k=1}^{g}(X - \alpha_k^2)^2.$$

Proof. It is shown in [22;5, Section 2] that in cases (a) and (b), respectively, Γ is a Schottky group of rank g over K. Put

$$s_0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \bmod(K^\times),$$

and for each $k = 1,\dots,g$, put

$$s_k = \begin{pmatrix} \lambda_k & \mu_k \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} \lambda_k & \mu_k \\ 1 & 1 \end{pmatrix}^{-1} \bmod(K^\times).$$

Then $s_0, s_1,\dots,s_g$ are of order 2, and for any $k = 1,\dots,g$,

$$\begin{aligned} s_k s_0 &= \begin{pmatrix} \lambda_k+\mu_k & 2\lambda_k\mu_k \\ 2 & \lambda_k+\mu_k \end{pmatrix} \bmod(K^\times) \\ &= \begin{pmatrix} \alpha_k(1+\beta_k^2) & \alpha_k^2(1-\beta_k^2) \\ 1-\beta_k^2 & \alpha_k(1+\beta_k^2) \end{pmatrix} \bmod(K^\times) \\ &= \gamma_k. \end{aligned}$$

Hence Γ is a Whittaker group in the terminology of [6, Chapter IX, 2.1]. Let Γ' be the subgroup of $PGL_2(K)$ generated by $s_0, s_1,\dots,s_g$, in which Γ is contained with index 2.

It is shown in [13, Section 7; 6, pp. 46–47] that in cases (a) and (b), respectively, for $z \in H_\Gamma - \bigcup_{\gamma\in\Gamma}\gamma(\infty)$,

$$\eta(z) = z \prod_{\gamma\in\Gamma-\{1\}} \frac{z-\gamma(0)}{z-\gamma(\infty)}$$

is convergent absolutely and uniformly in the wider sense, and hence $\eta(z)$ becomes a meromorphic function on H_Γ. For any $\delta\in\Gamma$, $\eta(\delta(z)) = \chi(\delta)\cdot\eta(z)$, where

$$\chi(\delta) = \left(-\frac{a}{d}\right) \prod_{\gamma\in\Gamma-\{1,\delta\}} \frac{a-c\gamma(0)}{a-c\gamma(\infty)} \quad \left(\delta := \begin{pmatrix} a & b \\ c & d \end{pmatrix} \bmod(K^\times)\right)$$

is independent of z and hence is multiplicative on δ. Since $\Gamma s_0 = s_0\Gamma$, we have

$$\eta(z) = z \prod_{\gamma\in\Gamma-\{1\}} \frac{z+\gamma(0)}{z+\gamma(\infty)},$$

which implies that $\eta(s_0(z)) = -\eta(z)$. Hence there is a character $\chi : \Gamma' \to K^\times$ such that $\eta(\delta(z)) = \chi(\delta)\cdot\eta(z)$ $(\delta\in\Gamma')$, and $\mathrm{Im}(\chi)\subset\{\pm1\}$ because Γ' is generated by the elements $s_0, s_1, \ldots, s_g$ of order 2. Thus $\theta(z) = \eta(z)^2$ is Γ'-invariant, and hence defines a meromorphic function on the quotient space H_Γ/Γ' with only one simple pole. Therefore, we have $\theta : H_\Gamma/\Gamma' \xrightarrow{\sim} \mathbf{P}^1_K$. Then as is shown in [6, p. 279], the fixed points of $s_0, s_1, \ldots, s_g$ belong to H_Γ and are ramification points of the natural covering $H_\Gamma/\Gamma \to H_\Gamma/\Gamma'$ of degree 2. Hence $C_\Gamma = H_\Gamma/\Gamma$ becomes a hyperelliptic curve over K, and its affine equation is given as above. The description of its degenerate form is derived from [9, Proposition 2.2] and that for any $\gamma \in \Gamma - \{1\}$,

$$\frac{z-\gamma(0)}{z-\gamma(\infty)} = 1 - \frac{\gamma(0)-\gamma(\infty)}{z-\gamma(\infty)} \to 1 \text{ under } \beta_1, \ldots, \beta_g \to 0. \qquad \square$$

2. Universal 1-forms and periods

Differential 1-forms and period integrals of Schottky uniformized curves were described by Schottky [22] and Manin–Drinfeld [14] (cf. [7,13]), and these universal expressions as power series were obtained in [8,9]. In this section, we give a hyperelliptic version of this result by using Theorem 1. Let x_k, y_k $(k=1,\ldots,g)$, p, and z be variables. Let A be the ring of formal power series over $\mathbf{Z}[1/2, x_1^{\pm1}, \ldots, x_g^{\pm1}, \prod_{i\neq j} 1/(x_i \pm x_j)]$ with variables $y_1, \ldots, y_g$, i.e.

$$A = \mathbf{Z}\left[\frac{1}{2}, x_1^{\pm1}, \ldots, x_g^{\pm1}, \prod_{i\neq j}\frac{1}{x_i \pm x_j}\right][[y_1, \ldots, y_g]],$$

and put

$$A_p = A\left[\prod_{k=1}^{g} \frac{1}{(x_k-p)(-x_k-p)}\right].$$

For each $k=1,\ldots,g$, let φ_k be the element of $PGL_2(\Omega)$ (Ω: the quotient field of A) given by

$$\varphi_k=\begin{pmatrix} x_k & -x_k \\ 1 & 1\end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & y_k^2\end{pmatrix}\begin{pmatrix} x_k & -x_k \\ 1 & 1\end{pmatrix}^{-1} \bmod(\Omega^\times),$$

and let Φ be the subgroup of $PGL_2(\Omega)$ with free generators $\varphi_1,\ldots,\varphi_g$. Let Φ_j (resp. Φ_{ij}) is a complete set of representatives of the cosets $\Phi/\langle\varphi_j\rangle$ (resp. $\langle\varphi_i\rangle\setminus\Phi/\langle\varphi_j\rangle$), and define the map $\psi_{ij}:\Phi_{ij}\to\Omega^\times$ by

$$\psi_{ij}(\varphi)=\begin{cases} y_i^2 & \text{if } i=j \text{ and } \varphi\in\langle\varphi_i\rangle \\ [x_i,-x_i;\varphi(x_j),\varphi(-x_j)] & \text{otherwise,}\end{cases}$$

where $[a,b;c,d]$ denotes $\{(a-c)(b-d)\}/\{(a-d)(b-c)\}$ as above. Then we define formally

$$\Omega_j=\sum_{\varphi\in\Phi_j}\left(\frac{\varphi(x_j)-\varphi(-x_j)}{(z-\varphi(x_j))(z-\varphi(-x_j))}\right)\mathrm{d}z \quad (j=1,\ldots,g),$$

$$W_{n,p}=\sum_{\varphi\in\Phi}\frac{\varphi'(z)}{(\varphi(z)-p)^n}\,\mathrm{d}z \quad (n\geq 1),$$

$$P_{ij}=\prod_{\varphi\in\Phi_{ij}}\psi_{ij}(\varphi) \quad (i,j\in\{1,\ldots,g\}).$$

Theorem 2. (a) Ω_j $(j=1,\ldots,g)$ *and* $W_{n,p}$ $(n\geq 1)$ *are* 1-*forms having power series expansions for* $z-p$ *with coefficients in* A_p, *and* P_{ij} $(i,j\in\{1,\ldots,g\})$ *belong to* A. *Moreover, we have the following congruences modulo the ideal generated by* $y_1^2,\ldots,y_g^2$:

$$\Omega_j\equiv\frac{2x_j}{z^2-x_j^2}\,\mathrm{d}z,\quad W_{n,p}\equiv\frac{1}{(z-p)^n}\,\mathrm{d}z,\quad P_{ij}\equiv\left(\frac{x_i-x_j}{x_i+x_j}\right)^2.$$

(b) *Assume that* $K=\mathbf{C}$, *let* $\alpha_k,\beta_k,\Gamma,C_\Gamma$ *be as in Theorem* 1(a), *and take* $p\in F_\Gamma-\{\infty\}$. *Then the coefficients of* $\Omega_j,W_{n,p}$ *and* P_{ij} *are absolutely convergent for* $x_k=\alpha_k, y_k=\beta_k$ $(k=1,\ldots,g)$. *Moreover,*

$$\omega_j=\Omega_j|_{x_k=\alpha_k,y_k=\beta_k} \quad (j=1,\ldots,g)$$

form a basis of differential 1-*forms of the first kind on* C_Γ *satisfying that*

$$\int_{a_i}\omega_j=2\pi\sqrt{-1}\,\delta_{ij} \quad (i=1,\ldots,g),$$

$$w_{n,p}=W_{n,p}|_{x_k=\alpha_k,y_k=\beta_k} \quad (n\geq 1)$$

become differential 1-*forms either of the second kind* (*if* $n>1$) *or of the third kind* (*if* $n=1$) *on* C_Γ *satisfying that*

$$\int_{a_i}w_{n,p}=0,\quad \int_{b_i}w_{1,p}=\int_\infty^p\omega_i \;\; (i=1,\ldots,g)$$

and

$$p_{ij} = P_{ij}|_{x_k=\alpha_k, y_k=\beta_k} \quad (i, j \in \{1, \ldots, g\})$$

become the multiplicative periods of $(C_\Gamma; a_i, b_i)$, *i.e.*

$$p_{ij} = \exp\left(\int_{b_i} \omega_j\right).$$

(c) *Assume that* K *is a nonarchimedean complete valuation field of characteristic* $\neq 2$, *let* $\alpha_k, \beta_k, \Gamma, C_\Gamma$ *be as in Theorem* 1(b), *and take* $p \in F_\Gamma - \{\infty\}$. *Then the coefficients of* $\Omega_j, W_{n,p}$ *and* P_{ij} *are absolutely convergent for* $x_k = \alpha_k,\ y_k = \beta_k$ $(k = 1, \ldots, g)$. *Moreover*,

$$\omega_j = \Omega_j|_{x_k=\alpha_k, y_k=\beta_k} \quad (j = 1, \ldots, g)$$

form a basis of differential 1-*forms of the first kind on* C_Γ,

$$w_{n,p} = W_{n,p}|_{x_k=\alpha_k, y_k=\beta_k} \quad (n \geq 1)$$

become differential 1-*forms either of the second kind* (*if* $n > 1$) *or of the third kind* (*if* $n = 1$) *on* C_Γ *and*

$$p_{ij} = P_{ij}|_{x_k=\alpha_k, y_k=\beta_k} \quad (i, j \in \{1, \ldots, g\})$$

become the multiplicative periods of C_Γ, *i.e. the Jacobian variety of* C_Γ *is isomorphic to the quotient* K-*analytic space of* $(K^\times)^g$ *by its subgroup with generators* $(p_{ij})_{1 \leq i \leq g}$ $(j = 1, \ldots, g)$.

Proof. Assertions (a) and (c) follow from Proposition 3.2 and Theorem 4.3 of [9], respectively by putting $x_{-k} = -x_k$ and $\alpha_{-k} = -\alpha_k$ $(k = 1, \ldots, g)$. Assertion (b) follows from classical Schottky uniformization theory in [22, Section 2] (cf. [7, Section 6; 13, Sections 7 and 8]). □

Remark. One can show that

$$\theta\left(x_k \frac{1 - y_k}{1 + y_k}\right), \quad \theta\left(x_k \frac{1 + y_k}{1 - y_k}\right) \in A \quad (k = 1, \ldots, g),$$

and that

$$Y^2 = X \prod_{k=1}^{g} \left(X - \theta\left(x_k \frac{1 - y_k}{1 + y_k}\right)\right) \left(X - \theta\left(x_k \frac{1 + y_k}{1 - y_k}\right)\right)$$

gives the affine equation of a hyperelliptic curve over $A[1/y_1, \ldots, 1/y_g]$ which is universal, i.e. becomes C_Γ under substituting $x_k = \alpha_k, y_k = \beta_k$ $(k = 1, \ldots, g)$ (see [11] for general case). Then the above Ω_j, $W_{n,p}$ and P_{ij} can be regarded as differential 1-forms and multiplicative periods of this universal hyperelliptic curve, respectively.

3. Hyperelliptic Jacobians

In this section, as is done in Theorem 3.2 and Corollary 3.3 of [8] for the (proper) Schottky problem, we give a solution to the hyperelliptic Schottky problem by using the universal periods P_{ij} in Theorem 2. For integers $g \geq 2$ and h, *Siegel modular forms* of degree g and weight h over a $\mathbf{Z}$-algebra R are defined as global sections of $\lambda^{\otimes h} \otimes_{\mathbf{Z}} R$ ($\lambda := \wedge^g \pi_*(\Omega_{\mathscr{A}/\mathscr{X}_g})$) on the moduli stack $\mathscr{X}_g$ of principally polarized abelian schemes of relative dimension g, where $\pi : \mathscr{A} \to \mathscr{X}_g$ is the universal abelian scheme. Then we recall the result of Chai and Faltings in [2–4] which says that to each Siegel modular form f of degree g and weight h over R, one can attach its (arithmetic) Fourier expansion

$$F(f) = \sum_{T=(t_{ij})} a(T) \prod_{i,j=1}^{g} q_{ij}^{t_{ij}} \in R[q_{ij}^{\pm 1}\ (i \neq j)][[q_{11}, \dots, q_{gg}]],$$

where q_{ij} ($i,j \in \{1,\dots,g\}$) are variables with symmetry $q_{ij} = q_{ji}$, and T runs through half-integral and positive semi-definite symmetric matrices of degree g. The Fourier expansion is functorial on R and becomes, when $k = \mathbf{C}$, the classical Fourier expansion with respect to $q_{ij} = \exp(2\pi\sqrt{-1}z_{ij})$ ($(z_{ij})_{1 \leq i,j \leq g} \in$ the Siegel upper half space of degree g). In the following, we give a characterization of the Fourier expansions of Siegel modular forms vanishing on the hyperelliptic Jacobian locus in $\mathscr{X}_g$, which consists of the Jacobian varieties of hyperelliptic curves with canonical polarization.

Theorem 3. *Let k be a field of characteristic $\neq 2$, and let f be a Siegel modular form of degree g and weight h over k. Then*

$$\begin{aligned} &f = 0 \text{ on the hyperelliptic Jacobian locus} \\ &\quad \Leftrightarrow F(f)|_{q_{ij}=P_{ij}} = 0 \text{ in } A \hat{\otimes}_{\mathbf{Z}} k. \end{aligned}$$

Proof. Take a nonarchimedean complete valuation field K containing k. Then by the construction of $F(f)$ (cf. [4, Chapter V]), for the periods p_{ij} given in Theorem 2(c), $F(f)|_{q_{ij}=p_{ij}}$ is (up to a canonical trivialization of $\lambda^{\otimes h}$) equal to the evaluation of f on the hyperelliptic curves C_Γ given in Theorem 1(b). Therefore, the implication ($\Rightarrow$) holds. On the other hand, as is shown in [6, pp. 282–284], any hyperelliptic Mumford curve of genus g over K can be uniformized by a Whittaker group which is by definition a Schottky group with free generators $t_1t_0, \dots, t_gt_0$, where $t_0, t_1, \dots, t_g \in PGL_2(K)$ are of order 2. Since

$$\rho t_0 \rho^{-1} = s_0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \bmod(K^\times)$$

for some $\rho \in PGL_2(K)$, C_Γ given in Theorem 1(b) form a Zariski dense subset in the moduli space of hyperelliptic curves of genus g. From this and the irreducibility of the moduli space, the implication ($\Leftarrow$) follows. $\square$

Remark. It would be possible and interesting to make an effective version of Theorem 3, i.e. to give an integer $n(g,h)$ explicitly described by g,h such that a Siegel modular form f of degree g and weight h over k vanishes on the hyperelliptic Jacobian locus if $F(f)|_{q_{ij}=P_{ij}} \in A\hat{\otimes}_{\mathbf{Z}}k$ belongs to the $n(g,h)$th power of the ideal generated by $y_1, \dots, y_g$.

By Theorem 3 and the congruence for P_{ij} given in Theorem 2(a), we have

Corollary. *Let f be a Siegel modular form of degree g over a field k of characteristic $\neq 2$, and denote its Fourier expansion by $\sum_{T=(t_{ij})} a(T)\prod_{i,j} q_{ij}^{t_{ij}}$. If $f=0$ on the hyperelliptic Jacobian locus, then for any set $\{s_1,\dots,s_g\}$ of nonnegative integers such that*

$$\sum_{i=1}^{g} s_i = \min\{\mathrm{tr}(T) \mid a(T)\neq 0\},$$

we have

$$\sum_{t_{ii}=s_i} a(T)\prod_{i<j}\left(\frac{x_i-x_j}{x_i+x_j}\right)^{4t_{ij}} = 0.$$

4. Solutions of KdV

In this section, as is done in Theorems 3.4 and 4.6 of [9] for the KP hierarchy, by using algebro-geometric theory on soliton equations (cf. [20,15]) and the universal 1-forms and periods given in Theorem 2, we construct formal and p-adic solutions of the KdV hierarchy given as the Lax form:

$$\frac{\partial L^2}{\partial t_{2n+1}} = [(L^{2n+1})_+, L^2]; \quad L^2 = \partial^2 + 2u(t_1,t_3,\dots)$$

($\partial = \partial/\partial t_1$, $(L^{2n+1})_+$: the nonnegative part of L^{2n+1} for ∂) which includes the KdV equation:

$$\frac{\partial u}{\partial t_3} - \frac{1}{4}\frac{\partial^3 u}{\partial t_1^3} - 3u\frac{\partial u}{\partial t_1} = 0.$$

First we treat the formal case. Let the notation be as in Section 2. For each $i=1,\dots,g$, define a square root $P_{ii}^{1/2}\in A$ of P_{ii} by

$$P_{ii}^{1/2} = y_i\sum_{n=0}^{\infty}\binom{1/2}{n}\left(\prod_{\varphi\in\Phi_{ii}-\{1\}}\psi_{ii}(\varphi)-1\right)^n$$

($\{1\}$ denotes the element of Φ_{ii} containing 1), and for any $\boldsymbol{v}=(v_i)_{1\le i\le g}\in\mathbf{Z}^g$, put

$$\prod_{i,j=1}^{g}(P_{ij})^{v_iv_j/2} = \prod_{i=1}^{g}(P_{ii})^{v_i^2/2}\prod_{i<j}(P_{ij})^{v_iv_j}.$$

Then for a sequence $\mathbf{w}=(w_i)_{1\le i\le g}$ and a vector $\boldsymbol{z}=(z_i)_{1\le i\le g}$ of indeterminates, the universal hyperelliptic theta function is defined by

$$\Theta(\mathbf{w}\cdot\exp(\boldsymbol{z})) = \sum_{\boldsymbol{v}\in\mathbf{Z}^g}\left\{\prod_{i,j=1}^{g}(P_{ij})^{v_iv_j/2}\prod_{i=1}^{g}w_i^{v_i}\sum_{n=0}^{\infty}\frac{1}{n!}\left(\sum_{i=1}^{g}v_iz_i\right)^n\right\},$$

which becomes a formal power series of $z_1,\dots,z_g$ over the ring

$$B = A[w_1^{\pm1},\dots,w_g^{\pm1}]\hat{\otimes}_{\mathbf{Z}}\mathbf{Q}.$$

In what follows, put $p=0$. Let $R_{jm}, Q_{nm} \in A$ $(j=1,\dots,g;\ m,n \in \mathbf{N})$ such that

$$\Omega_j = \sum_{m=1}^{\infty} R_{jm} z^{m-1}\,\mathrm{d}z,$$

$$W_{n+1,0} = \left(\frac{1}{z^{n+1}} + \sum_{m=1}^{\infty} \frac{Q_{nm}}{n} z^{m-1} \right) \mathrm{d}z,$$

and put $\boldsymbol{R}_m = (R_{jm})_{1 \le j \le g}$.

Theorem 4. *The formal power series*

$$u(t_1, t_3, \dots) = \frac{\partial^2}{\partial t_1^2} \log \Theta \left(\mathbf{w} \cdot \exp\left(\sum_{n=0}^{\infty} t_{2n+1} \boldsymbol{R}_{2n+1} \right) \right) + Q_{11}$$

of $t_1, t_3, \dots$ *over* B *satisfies the KdV hierarchy.*

Proof. Take $\alpha_k, \beta_k \in \mathbf{C}^\times$ as in Theorem 1(a), and let $\beta'_k \in \{\pm\beta_k\}$ such as

$$\beta'_k \sum_{n=0}^{\infty} \binom{1/2}{n} \left(\frac{p_{kk}}{\beta_k^2} - 1 \right)^n = \exp\left(\frac{1}{2} \int_{b_k} \omega_k \right).$$

Then by Theorem 2(b),

$$\Theta(\mathbf{w} \cdot \exp(\boldsymbol{z}))|_{x_k=\alpha_k, y_k=\beta'_k}$$

is the Riemann theta function of C_Γ. Further, $\boldsymbol{R}_m = \mathbf{0}$ if m is even because

$$\frac{1}{-z-\varphi(x_j)} - \frac{1}{-z-\varphi(-x_j)} = \frac{1}{z-\iota(\varphi)(x_j)} - \frac{1}{z-\iota(\varphi)(-x_j)}$$

for any $\varphi \in \Phi$, where ι is the involutive automorphism of Φ sending φ_k to φ_k^{-1}. Thus by results of Novikov [20] and McKean–van Moerbeke [15] (cf. [12,19]), for any $c_1, \dots, c_g \in \mathbf{C}^\times$,

$$u(t_1, t_3, \dots)|_{x_k=\alpha_k, y_k=\beta'_k, w_k=c_k}$$

satisfies KdV. Therefore, $u(t_1, t_3, \dots)$ itself satisfies KdV. $\square$

Remark. By Theorem 2(a), replacing $\mathbf{w}=(w_i)_i$ by $(w_i P_{ii}^{-1/2})_i$ as is done in [19, Chapter IIIb, Section 5] for the Riemann theta functions, one can see that $u(t_1, t_3, \dots)$ gives a deformation, as a solution of KdV, of the g-soliton solution

$$\frac{\partial^2}{\partial t_1^2} \log \left[1 + \sum_{\emptyset \neq I \subset \{1,\dots,g\}} \left\{ \prod_{i,j \in I;\ i<j} \left(\frac{x_i - x_j}{x_i + x_j} \right)^2 \prod_{i \in I} w_i \exp\left(-2 \sum_{n=0}^{\infty} \frac{t_{2n+1}}{x_i^{2n+1}} \right) \right\} \right]$$

(see [9, 3.5] for the KP case).

Second we treat the p-adic case. Let K be a nonarchimedean complete valuation field, and let C_Γ be a hyperelliptic curve over K as in Theorem 1(b), of which 1-forms

ω_j, $w_{n,p}$ and periods p_{ij} are given in Theorem 2(c). In what follows, we assume that K is of characteristic 0 and that

$$|\beta_k|^2 < \min\{|4[\alpha_k, -\alpha_k; \pm\alpha_i, \pm\alpha_j]|;\ i,j \neq k\} \quad (k = 1,\ldots,g)$$

(the latter condition is automatically satisfied if the residual characteristic of K is not 2). Then as is shown in [9, Theorem 4.3(c)], for any $i = 1,\ldots,g$,

$$\left|\frac{1}{4}\left(\frac{p_{ii}}{\beta_i^2} - 1\right)\right| < 1,$$

and hence

$$\beta_i \left\{\sum_{n=0}^{\infty} \binom{-1/2}{n} \left(\frac{p_{ii}}{\beta_i^2} - 1\right)^n\right\}^{-1} = \beta_i \left\{\sum_{n=0}^{\infty} \binom{2n}{n} \left(-\frac{1}{4}\right)^n \left(\frac{p_{ii}}{\beta_i^2} - 1\right)^n\right\}^{-1}$$

is convergent and becomes a square root of p_{ii} which we denote by $p_{ii}^{1/2}$. Then by the negative definiteness shown in [14, Section 4] of the form $\log|\prod_{i,j=1}^{g}(p_{ij})^{v_i v_j}|$ for $\boldsymbol{v} = (v_i)_{1\leq i\leq g} \in \mathbf{Z}^g$, one can see that for $\mathbf{c} = (c_i)_{1\leq i\leq g} \in (K^\times)^g$,

$$\Theta(\mathbf{c}\cdot\exp(z)) = \sum_{\mathbf{v}\in\mathbf{Z}^g} \left\{\prod_{i,j=1}^{g} (p_{ij})^{v_i v_j/2} \prod_{i=1}^{g} c_i^{v_i} \sum_{n=0}^{\infty} \frac{1}{n!} \left(\sum_{i=1}^{g} v_i z_i\right)^n\right\}$$

belongs to $K[[z_1,\ldots,z_g]]$. Let p=0 which defines a Weierstrass point of C_Γ by Theorem 1(c). Let $r_{jm}, q_{nm} \in K$ $(j = 1,\ldots,g;\ m,n \in \mathbf{N})$ such that

$$\omega_j = \sum_{m=1}^{\infty} r_{jm} z^{m-1}\, \mathrm{d}z,$$

$$w_{n+1,0} = \left(\frac{1}{z^{n+1}} + \sum_{m=1}^{\infty} \frac{q_{nm}}{n} z^{m-1}\right) \mathrm{d}z.$$

and put $\boldsymbol{r}_m = (r_{jm})_{1\leq j\leq g}$.

Theorem 5. *For any* $\mathbf{c} \in (K^\times)^g$,

$$u(t_1, t_3, \ldots) = \frac{\partial^2}{\partial t_1^2} \log \Theta\left(\mathbf{c}\cdot\exp\left(\sum_{n=0}^{\infty} t_{2n+1}\boldsymbol{r}_{2n+1}\right)\right) + q_{11} \in K[[t_1, t_3, \ldots]]$$

satisfies the KdV hierarchy.

Proof. This follows from Theorems 2(c) and 4. □

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